Four

STATIC PERTURBATION THEORY

4-3. Exact Perturbation Theory

* The perturbation theory approach can be expanded by including higher order perturbation terms.
  - The result would converge to the exact solution.
  - The resulting method is cumbersome and generally not practical.

* Derivation of Exact perturbation theory
  - The perturbed & unperturbed problem (modified by adjoint eqs.)

$$\mathbf{M}\Phi = \lambda \mathbf{F}\Phi \qquad (4.37a)$$

$$\mathbf{M}_0^*\Phi_0^* = \lambda_0 \mathbf{F}_0^*\Phi_0^* \qquad (4.37b)$$

  - These equations are multiplied by Φ_0^* and Φ,

$$\left(\Phi_0^*, \mathbf{M}\Phi\right) = \lambda\left(\Phi_0^*, \mathbf{F}\Phi\right) \qquad (4.38a)$$

$$\left(\Phi, \mathbf{M}_0^*\Phi_0^*\right) = \lambda_0\left(\Phi, \mathbf{F}_0^*\Phi_0^*\right) \qquad (4.38b)$$

  - The equation in Eq.(4.38b) is revolved as,

$$\left(\Phi, \mathbf{M}_0^*\Phi_0^*\right) = \lambda_0\left(\Phi, \mathbf{F}_0^*\Phi_0^*\right) = \lambda_0\left(\Phi_0^*, \mathbf{F}_0\Phi\right)$$

$$= \lambda_0\left(\Phi_0^*, \mathbf{F}\Phi\right) - \lambda_0\left(\Phi_0^*, \Delta\mathbf{F}\Phi\right) \qquad (4.38d)$$

  - Therefore, solving for $-\Delta\lambda = \Delta\rho$ with subtracting Eq.(4.38a)and (4.38d) yields,

$$\Delta\rho = \frac{\left(\Phi_0^*, \left[\lambda_0\Delta\mathbf{F} - \Delta\mathbf{M}\right]\Phi\right)}{\left(\Phi_0^*, \mathbf{F}\Phi\right)} = -\Delta\lambda \qquad (4.39)$$

* The exact perturbation formula for reactivity increments
  The first-order perturbation formula is obtained as a first approximation of the exact formula by approximating Φ with Φ_0 and $\mathbf{F}$ with $\mathbf{F}_0$.
  The errors of first-order perturbation are 2nd order.

* Exact formula for the reactivity
  - If Φ_0 describes a specific critical state, $\lambda_0 = 1$.

$$\rho = \rho^{st} = \frac{\left(\Phi_0^*, \left[\Delta\mathbf{F} - \Delta\mathbf{M}\right]\Phi\right)}{\left(\Phi_0^*, \mathbf{F}\Phi\right)} \qquad (4.40)$$

- The reactivity(ρ) of a state with flux Φ is independent of the specifics of the critical reference state. → the $\mathbf{F}_0$ and $\mathbf{M}_0$ terms in Eq. (4.40) cancel.

$$\left(\Phi_0^*, \left[\mathbf{F}_0 - \mathbf{M}_0\right]\Phi\right) = 0 \qquad (4.41)$$

- The weighting function need not be the adjoint of any critical problem.

$$\rho^{st} = \frac{\left(\Phi^w, \left[\mathbf{F} - \mathbf{M}\right]\Phi\right)}{\left(\Phi^w, \mathbf{F}\Phi\right)} \qquad (4.42)$$

- If the state is near a known critical state with the adjoint flux Φ_0^*, it can be numerically advantageous to choose this adjoint flux as a weighting function.

$$\rho = \rho^{st} = \frac{\left(\Phi_0^*, \left[\mathbf{F} - \mathbf{M}\right]\Phi\right)}{\left(\Phi_0^*, \mathbf{F}\Phi\right)} \qquad (4.43)$$

* Application of the exact perturbation formula provide a tool to improve the accuracy of the reactivity.
  - It indicates how to improve practically on the first-order theory results without going to the formal but impractical route of higher order perturbation theories.

4-4 Application of First-order Perturbation Theory

* To illustrate the application of the 1st-order perturbation formula
* To demonstrate the effect of the adjoint flux weighting

(Applications of 1st-order and exact perturbation theories →Chap. 9 & 11)

* Simple one-group model
  - The neutronics eigenvalue problem is self-adjoint → $\Phi^* = \Phi$
  - The flux is just a single number → Φ_0, Φ_0^* cancel.
  - The formula for the reactivity increment :

$$\Delta\rho^{(1)} = \frac{\lambda_0 \Delta\nu\Sigma_f - \Delta\Sigma_a'}{\nu\Sigma_{f0}} = \frac{\Sigma_{a0}'\Delta\nu\Sigma_f}{\left(\nu\Sigma_{f0}\right)^2} - \frac{\Delta\Sigma_a'}{\nu\Sigma_{f0}} \qquad (4.45)$$

,where $\Sigma_{a0}' = D_0 B_0^2 + \Sigma_{a0}$ (4.46)

- Reference value of ρ

$$\rho_0 = 1 - \frac{1}{k_0} = \frac{\nu\Sigma_{f0} - \Sigma'_{a0}}{\nu\Sigma_{f0}} \qquad (4.47)$$

The first order variation of Eq.(4.47) is,

$$\delta\rho^{(1)} = \delta\left(1 - \frac{\Sigma'_{a0}}{\nu\Sigma_{f0}}\right) = \frac{\Sigma'_{a0}\delta\nu\Sigma_f}{\left(\nu\Sigma_{f0}\right)^2} - \frac{\delta\Sigma'_{a0}}{\nu\Sigma_{f0}} \qquad (4.48)$$

* This agrees with Eq.(4.45b)
→Exact reactivity increment can be calculated,
→No need to apply a first-order approximation

* Space-dependent perturbation due to a local increase in absorption (e.g. CR insertion)

- 1-D, slab reactor (x-direction) model
- Unperturbed criticality eigenvalue problem for a spatially uniform composition

$$\left(-D_0\frac{d^2}{dx^2} + \Sigma_{a0}\right)\phi_0(x) = \lambda_0\nu\Sigma_{f0}\phi_0(x) \qquad (4.49)$$

,with $\phi_0(x_b) = 0$ at $x_b = \pm a$ (4.50)

- Space-dependent flux as determined by the boundary conditions.

$$\phi_0(x) = \phi_0 \cos Bx \qquad (4.51)$$

,where $B^2 = \left(\frac{\pi}{2a}\right)^2$ (geometrical buckling) (4.52)

- Insert Eq.(4.51) into Eq.(4.49)

$$\frac{1}{\lambda_0} = k_0 = \frac{\nu\Sigma_f}{D_0B^2 + \Sigma_{a0}} = \frac{\nu\Sigma_{f0}}{\Sigma'_{a0}} \qquad (4.53)$$

- If the change consists only of a change in Σ_a (e.g. $\Delta\nu\Sigma_f = 0$) in a range from $-x_p$ to $+x_p$, the perturbation reactivity in the one-group approximation ($\phi_0^*(x) = \phi_0(x)$) is given,

$$\Delta\rho^{(1)} = \frac{-\int_{-x_p}^{x_p}\phi_0(x)\Delta\Sigma_a\phi_0(x)dx}{\int_{-a}^{a}\phi_0(x)\nu\Sigma_{f0}\phi_0(x)dx} \qquad (4.55)$$

The integral in the denominator is,

$$\phi_0^2\nu\Sigma_{f0}\int_{-a}^{a}\cos^2 Bxdx = \nu\Sigma_{f0}\cdot a\phi_0^2 \qquad (4.56)$$

If x_p is small, the cosine in the numerator integral can be unity. Then the numerator of Eq.(4.55) becomes :

$$-\phi_0^2 \int_{-x_p}^{x_p} \Delta\Sigma_a \cos^2 Bxdx \cong -\phi_0^2 \Delta\Sigma_a \cdot 2x_p \qquad (4.57)$$

- It yields the first-order perturbation reactivity :

$$\Delta\rho^{(1)} \cong -2 \cdot \frac{\Delta\Sigma_a}{\nu\Sigma_{f0}} \cdot \frac{x_p}{a} \qquad (4.58)$$

, where $-\Delta\Sigma_a / \nu\Sigma_{f0}$: homogeneous change in Σ_a

x_p / a : the relative lengths of domains

- Change in Σ_a over entire separated homogeneous reactor composition ($x_p = a$) :

$$\Delta\rho^{(1)} = -\frac{\Delta\Sigma_a}{\nu\Sigma_{f0}} \qquad (4.59)$$

- Effect of adjoint weighting

If Omitted,

$$\Delta\rho^{(1)} = \frac{-\int_{-x_p}^{x_p} \Delta\Sigma_a \phi_0(x)\,dx}{\int_{-a}^{a} \nu\Sigma_{f0}\phi_0(x)\,dx} \qquad (4.60)$$

If $x_p = a$, the adjoint weighting has no influence. But, for a perturbation in a small change,

$$\Delta\rho^{(0)} \cong -\frac{\pi}{2} \cdot \frac{\Delta\Sigma_a}{\nu\Sigma_{f0}} \cdot \frac{x_p}{a} \qquad (4.61)$$

- This accounts for only a part of the total effect. The inclusion of the adjoint weighting shows that the importance of the center of perturbation is more pronounced